International Journal of Theoretical Physics, Vol. 28, No. 7, 1989

Classification of the Even–Even Nuclei in Symplectic Multiplets

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Received January 13, 1989

A classification of the even-even nuclei with $Z \geq 20$, $A-Z \geq 20$, in terms of the boson representation of the $sp(4, R)$ algebra is proposed. All even-even nuclei whose valence nucleons occupy the same major nuclear shell are united in two symplectic multiplets and thus treated in a unified way. A qualitative analysis of the spectrum of the 2^+ energy levels of the ground (quasiground) bands is carried out. This analysis shows the expediency of the classification scheme proposed—a periodic structure of the same type is observed in the different shells. This periodic structure is especially stable in the case of the heavy and superheavy nuclei.

1. INTRODUCTION

The introduction of the F spin in the framework of IBM-2 (Arima *et al.* 1977) has inspired the idea of considering in a unified way the properties of sequences of atomic nuclei. Thus, Harter *et al.* (1985) and von Brentano *et al.* (1985) classify series of even-even nuclei in F-spin multiplets. The empirical analysis carried out in these papers reveals the advantages of this classification. This analysis shows that the low-lying energy levels of the ground and gamma bands of the nuclei of a given F-spin multiplet depend slightly, almost constantly, on the third projection of the F spin.

In the present paper we generalize this approach by proposing a classification scheme within which all even-even nuclei whose valence nucleons belong to a given major shell are united in two symplectic multiplets. This enables us to treat in a unified way the entire spectrum for each shell, which allows us to reveal both existing regularities and the typical features of the different shells.

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0020-7748/89/0700-0769$06.00/0

In order to clarify the classification problem under consideration, it is useful to introduce the concept of the generalized dynamical group (GDG). By a dynamical group (DG) we mean, as usual (Dashen and Gell-Mann, 1965; Bohm and Barut, 1965; Dothan *et al.*, 1965*a*), a group which gives the actual energy of a quantum mechanical system. In the case of the application of the DG concept to the description of the collective nuclear properties, one appropriately chosen irreducible representation of the DG gives the entire spectrum of the collective states of a given nucleus. By a GDG we mean a group beyond DG. One irreducible representation of GDG gives the entire spectrum of the collective states not of one, but of a sequence of nuclei. In other words, by means of GDG, sequences of nuclei together with their collective states are united in common multiplets. It is evident that

$$\mathrm{GDG} \supset \mathrm{DG}$$

In this way the introduction of GDG leads to the description of the energy spectra of series of nuclei in a unified way, i.e., by means of a common Hamiltonian, whose coefficients are the same for a given sequence of nuclei.

Different candidates for DG as a group for the description of the collective states of the even–even nuclei have been proposed in the literature since the pioneering work of Elliott (1958), who was the first to investigate the role of $SU(3)$ for the description of light nuclei. In particular we mention:

$SL(3, R)$ (Weaver and Biedenharn, 1970).

$Sp(6, R)$ (Raychev, 1972; Afanasjev and Raychev, 1972; Rosensteel and Rowe, 1976).

$SU(3)$ (Ratna Raju *et al.*, 1973; Raychev and Roussev, 1978).

$Sp(12, R)$ (Vanagas *et al.*, 1975; Heyde *et al.*, 1984).

$U(6)$, IBM-1 (Arima and Iachello, 1975; Janssen *et al.*, 1974; Kyrchev, 1980).

$U(6)$, interacting vector boson model (Georgieva *et al.*, 1982).

$U(6)\otimes U(6)$, IBM-2 (Arima *et al.*, 1977).

In the common case, the set of collective solutions is infinite and respectively DG is noncompact, i.e., the irreducible representations of DG are infinite. Sometimes the problem can be approximated by finite sets of solutions and the DG should be compact. The question of the choice of DG is still open.

As mentioned above, in the case of IBM-2 the dynamical group is $\mathrm{DG} \equiv U_\pi(6)\otimes U_\nu(6)$ [the notations are from Elliott (1985)]. When the boson number N is fixed, the different irreducible unitary representations (IURs) of $U_\pi(6)\otimes U_\nu(6)$, corresponding to different nuclei, are labeled by the third projection F_0 of the F spin. The direct sum of the spaces of these representations coincides with the space of one most symmetric representation of the group $U(12)$ labeled by N. Thus, sequences of nuclei together with their collective states can be united in common multiplets and the group $U(12)$ (Elliott, 1985; Frank and van Isacker, 1985; Solari *et al.*, 1987) arises as a GDG.

Now the problem is to generalize this scheme so that the even–even nuclei, whose valence nucleons occupy the same major nuclear shell, can be treated in a unified way. One possible way is to extend $U(12)$ to the symplectic $Sp(24, R)$, i.e., to consider $Sp(24, R)$ as a GDG. This possibility is discussed in Sections 2 and 3. In particular, in Section 2, the algebraic construction of the extension $U(12)\to Sp(24, R)$ is given. In Section 3 a classification scheme is introduced. According to this scheme the even–even nuclei with valence nucleons belonging to a given major shell are united in two $Sp(24, R)$ multiplets. However, as shown in Section 3, the consecutive realization of this extension leads to some difficulties. Thus, there arises an asymmetry when the collective states of nuclei, similar in their nature, are described by means of IURs of the $\mathrm{DG}\equiv U_\pi(6)\otimes U_\nu(6)$ that essentially differ in their dimensions. There appear unphysical states, which requires the introduction of a proper selection rule.

In Section 4 an alternative approach is proposed, which holds if, neglecting for the time being the problem of the description of the collective nuclear states, one concentrates only on the problem of the classification of the nuclei. This approach is based on the group $Sp(4, R)$ as a nuclear classification group (CG). In this case the even–even nuclei are again united in two $Sp(4, R)$ multiplets arranged in the same order as in the $Sp(24, R)$ scheme. As for the description of the collective states, we suppose that

$$\begin{array}{c} \mathrm{GDG}\supset \mathrm{CG}\otimes\mathrm{DG} \\ \quad\;\; \| \\ \quad\;\; Sp(4, R) \end{array}$$

Here we do not fix the groups DG and GDG, respectively—the problem of their proper choice is beyond the purpose of this paper. The important point is that now there is no asymmetry, which appears in the $Sp(24, R)$ scheme. The members of each $Sp(4, R)$ multiplet are uniquely determined by their mass number A and charge Z. That is why the energy spectrum of the multiplet as a whole should depend on these quantitites.

In Section 5 the $Sp(4, R)$ multiplets corresponding to the major nuclear shells at $A\geq 40$ are discussed. A qualitative analysis of the spectrum of the 2^+ ground and quasiground levels is carried out. This analysis shows the expediency of the classification scheme proposed—a periodic structure of one and the same type is observed in the different shells. This periodic structure is especially stable in the case of the heavy and superheavy nuclei.

2. ALGEBRAIC CONSTRUCTION OF THE EXTENSION $U(12) \to Sp(24, R)$

In IBM-2 two types of boson creation (π_a^+ and ν_a^+) and annihilation (π_a and ν_a) operators ($a = 0, 1, \ldots, 5$) are introduced. The bilinear products $\pi_a^+\pi_b$ and $\nu_a^+\nu_b$ generate the "proton" and "neutron" $U(6)$ groups, i.e., $U_\pi(6)$ and $U_\nu(6)$. The introduction of the operators $\pi_a^+\nu_b$ and $\nu_a^+\pi_b$ extends the $u_\pi(6)\oplus u_\nu(6)$ algebra to $u(12)$. With the help of boson operators one can define only the most symmetric representations of $u_\pi(6)$, $u_\nu(6)$, and $u(12)$ labeled by N_π, N_ν, and $N = N_\pi + N_\nu$, respectively. From the generators of $U(12)$ one can construct the sums $\pi_a^+\pi_b + \nu_a^+\nu_b$, which generate the "mixed" $U_{\pi\nu}(6)$ group, and also the operators

$$F_+ = \sum_{a=0}^{5} \pi_a^+\nu_a, \qquad F_- = \sum_{a=0}^{5} \nu_a^+\pi_a, \qquad F_0 = \tfrac{1}{2}(N_\pi - N_\nu)$$

(where $N_\pi = \sum_{a=0}^{5} \pi_a^+\pi_a$ and $N_\nu = \sum_{a=0}^{5} \nu_a^+\nu_a$), which generate the F-spin group $SU_F(2)$. This corresponds to the decomposition $U(12) \supset U_{\pi\nu}(6)\otimes SU_F(2)$.

The extension of $u(12)$ to $sp(24, R)$ can be done in a natural way [the common case of $sp(4k, R)$ is discussed in Georgieva *et al.* (1985)]. The boson representation of $sp(24, R)$ (Itsykson, 1967) is obtained by the addition of raising ($\pi_a^+\pi_b^+$, $\nu_a^+\nu_b^+$, $\pi_a^+\nu_b^+$) and decreasing ($\pi_a\pi_b$, $\nu_a\nu_b$, $\pi_a\nu_b$) operators to the generators of $U(12)$. All most symmetric representations of $u(12)$ labeled by N act in spaces whose direct sum coincides with the space $\mathscr{H}$ of the boson representation of $sp(24, R)$. The latter is reducible and decomposes into two irreducible ones. The first acts in the space $\mathscr{H}_+$, where the spectrum of N is even, and the second acts in the space $\mathscr{H}_-$, where N is odd ($\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$).

The groups $SU_F(2)$ and $U_{\pi\nu}(6)$ are mutually complementary (Moshinsky and Quesne, 1971), which leads to the following relation for their second-order Casimir operators: $C_2^{(6)} = 2\mathbf{F}^2 + 4N + \frac{1}{2}N^2$. Hence, when N is fixed, the eigenvalues $F(F+1)$ of $\mathbf{F}^2$ give the IURs of both $SU_F(2)$ and $U_{\pi\nu}(6)$. Further, it is obvious that when N and F are fixed, there arise $2F+1$ equivalent representations of $U_{\pi\nu}(6)$ labeled by $F_0 = -F, \ldots, F$. Thus, one obtains the following reduction scheme:

$$sp(24, R) \xrightarrow{N} u(12) \xrightarrow{\mathbf{F}^2} su_F(2)\oplus u_{\pi\nu}(6) \xrightarrow{F_0} u_{\pi\nu}(6) \qquad (1)$$

On the other hand, in the space $\mathscr{H}$ there acts a reducible unitary representation, namely the ladder representation, of the algebra $u(6, 6)$ (Dothan *et al.*, 1965*b*; Todorov, 1966). The corresponding Weyl generators of $U(6, 6)$ are

$$\pi_a^+\pi_b, \quad \pi_a^+\nu_b^+, \quad -\nu_a\pi_b, \quad -\nu_a\nu_b^+$$

This representation splits into irreducible ones (ladders), labeled by the first-order Casimir operator of $U(6,6)$:

$$C_1^{(6,6)} = 2F_0 - 6$$

In the space of each ladder (F_0 fixed) there acts an infinite set of IURs of the algebra $u_\pi(6) \oplus u_\nu(6)$ (steps) labeled by N. The reduction

$$u_\pi(6) \oplus u_\nu(6) \rightarrow u_{\pi\nu}(6)$$

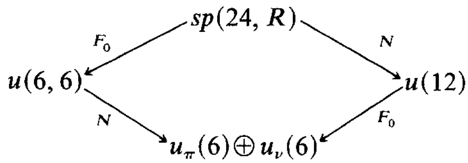
can be obtained by means of $\mathbf{F}^2(C_2^{(6)})$. Finally, instead of (1), one has

$$sp(24, R) \xrightarrow{F_0} u(6,6) \xrightarrow{N} u_\pi(6) \oplus u_\nu(6) \xrightarrow{\mathbf{F}^2} u_{\pi\nu}(6) \qquad (2)$$

Reduction schemes (1) and (2) are written in terms of algebras. We recall that the IURs of the group $U(n)$ and the corresponding IURs of the algebra $u(n)$ act in the same spaces.

The general case $sp(2dn, R) \rightarrow u(p,q) \oplus u(n)$, $p+q=d$, was investigated by Quesne (1986); when $n=1$, $p=q=k$ ($d=2k$) this reduction can be written as $sp(4k, R) \rightarrow u(k,k)$. From a mathematical point of view both schemes (1) and (2) are equally appropriate for the description of all IURs of $u_{\pi\nu}(6)$ acting in $\mathcal{H}$.

The splitting of the spaces $\mathcal{H}_\pm$ corresponding to the reductions



is shown schematically in Figure 1, where the columns represent the ladders defined by F_0 and the rows represent the IURs of $u(12)$ defined by N. Each cell corresponds to a given IUR of $u_\pi(6) \oplus u_\nu(6)$.

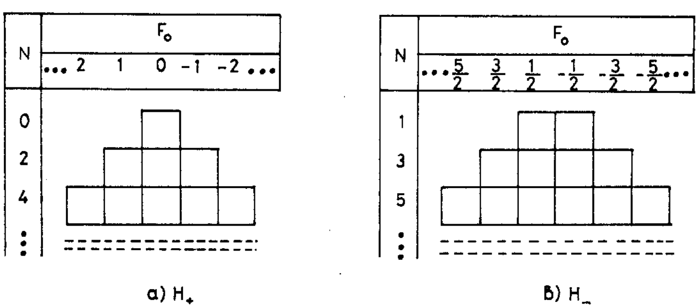


Fig. 1. The splitting of $\mathcal{H}_+$ (N even) and $\mathcal{H}_-$ (N odd) corresponding to the reductions $sp(4k, R) \rightarrow u(k,k) \rightarrow u(k) \oplus u(k)$ and $sp(4k, R) \rightarrow u(2k) \rightarrow u(k) \oplus u(k)$, $k = 1, 6$.

3. CLASSIFICATION SCHEME BASED ON THE EXTENSION $U(12) \to Sp(24, R)$

In IBM-2 the proton and neutron boson numbers N_π and N_ν are found by counting the valence proton and neutron pairs (or hole pairs) of a given nucleus from the nearest closed shell. The quantities N and F_0 (see, for instance, Elliott 1985) are defined by

$$N = N_\pi + N_\nu, \qquad F_0 = \tfrac{1}{2}(N_\pi - N_\nu) \tag{3}$$

In various papers dealing with IBM-2 the following four possibilities to count N_π and N_ν are used:

(i) From proton and neutron particles. In this case one has

$$N_\pi = \tfrac{1}{2}(N_p - N_p^{\text{mag}}), \qquad N_\nu = \tfrac{1}{2}(N_n - N_n^{\text{mag}}) \tag{4}$$

where N_p and N_n are the total proton and neutron numbers of the nucleus and N_p^{mag} and N_n^{mag} are the corresponding magic numbers. Therefore

$$N = \tfrac{1}{2}(A - A^{\text{mag}}), \qquad F_0 = \tfrac{1}{2}(M_T - M_T^{\text{mag}}) \tag{5}$$

where $A = N_p + N_n$ is the mass number and $M_T = \tfrac{1}{2}(N_p - N_n)$ is the third projection of the isospin.

(ii) From proton and neutron holes. Then

$$N = \tfrac{1}{2}(A^{\text{mag}} - A), \qquad F_0 = \tfrac{1}{2}(M_T^{\text{mag}} - M_T) \tag{6}$$

and the difference between this case and the previous one is not significant.

(iii) From proton particles and neutron holes. Then

$$N = M_T - M_T^{\text{mag}}, \qquad F_0 = \tfrac{1}{4}(A - A^{\text{mag}})$$

(iv) From proton holes and neutron particles. Then

$$N = M_T^{\text{mag}} - M_T, \qquad F_0 = \tfrac{1}{4}(A^{\text{mag}} - A)$$

We do not stick to the interpretation of N_π and N_ν as numbers of real pair excitations in nuclei. The physical sense of N and F_0 is revealed by their expressions in terms of A and M_T. From this point of view it is evident that compared with cases (i) and (ii) the physical meaning of N and F_0 in cases (iii) and (iv) is exchanged. But in order to describe the even–even nuclei in a unified way a uniqueness in the understanding of N and F_0 is necessary. Moreover, if we want to introduce a classification scheme according to which the even–even nuclei from a given major shell are united in common multiplets, then it is not acceptable to assume that for the first half of the shell N and F_0 are given by (5) and for the second half by (6). In our opinion the most natural way to count N_π and N_ν is given by (4). Then N and F_0 are defined by (5) and the even–even nuclei from a given major nuclear shell are enumerated by the values of the pair (N, F_0).

Table I. Multiplet $(20, 20|28, 28)_-$

	F_0		
N	1/2	−1/2	−3/2
1	^{42}Ti	^{42}Ca	
3	^{46}Cr	^{46}Ti	^{46}Ca
5	^{50}Fe	^{50}Cr	^{50}Ti
7	^{54}Ni	^{54}Fe	

A major nuclear shell is defined by a pair of two double magic numbers (N'_p, N'_n) and (N''_p, N''_n), where $N'_p < N''_p$ and $N'_n < N''_n$. The even–even nuclei whose valence nucleons belong to this shell can be united in two symplectic multiplets in the following way.

The double magic number (N'_p, N'_n) corresponds to the vacuum state $(N=0)$ in $\mathscr{H}$. Using formulas (4) and (5), one finds N_π and N_ν, and respectively N and F_0. Then each nucleus corresponds to a definite cell in the space $\mathscr{H}_+$ or $\mathscr{H}_-$, which represents a given IUR of $u_\pi(6) \oplus u_\nu(6)$ (see Figure 1). The symplectic multiplets obtained in this way will be denoted by $(N'_p, N'_n|N''_p, N''_n)_+$ if N is even and by $(N'_p, N'_n|N''_p, N''_n)_-$ if N is odd. In $\mathscr{H}_+$ and $\mathscr{H}_-$ these multiplets form closed figures restricted by the conditions $0 \leq N_\pi \leq \frac{1}{2}(N''_p - N'_p)$ and $0 \leq N_\nu \leq \frac{1}{2}(N''_n - N'_n)$, so that $0 \leq N \leq \frac{1}{2}(A'' - A')$. In other words, the space of the even–even nuclei whose valence nucleons belong to a given major shell is mapped onto two finite subspaces of $\mathscr{H}_+$ and $\mathscr{H}_-$, respectively. Within these figures the spectrum of F_0 is also restricted: $\frac{1}{4}(N'_n - N''_n) \leq F_0 \leq \frac{1}{4}(N''_p - N'_p)$. This quantity runs over all its admissible values $F_0 = -N/2, \ldots, N/2$ if and only if $N \leq \frac{1}{2}(N''_n - N'_n)$ and $N \leq \frac{1}{2}(N''_p - N'_p)$. The sides of the figures correspond to closed neutron or proton shells. Each row includes nuclei belonging to a given isobar, and each column includes nuclei belonging to a given isofer. Tables I–VIII are

Table II. Multiplet $(20, 28|28, 50)_-$

	F_0		
N	3/2	1/2	−1/2
1		^{50}Ti	^{50}Ca
3	^{54}Fe	^{54}Cr	^{54}Ti
5	^{58}Ni	^{58}Fe	^{58}Cr
7		^{62}Ni	^{62}Fe
9			^{66}Ni

Table III. Multiplet (28, 28|50, 50)$_-$

N	F_0: $-1/2$	$-3/2$	$-5/2$	$-7/2$	$-9/2$
1	^{58}Ni				
3	^{62}Zn	^{62}Ni			
5	^{66}Ge	^{66}Zn	^{66}Ni		
7	^{70}Se	^{70}Ge	^{70}Zn		
9	^{74}Kr	^{74}Se	^{74}Ge	^{74}Zn	
11	^{78}Sr	^{78}Kr	^{78}Se	^{78}Ge	^{78}Zn
13	^{82}Zr	^{82}Sr	^{82}Kr	^{82}Se	^{82}Ge
15		^{86}Zr	^{86}Sr	^{86}Kr	
17		^{90}Mo	^{90}Zr		
19		^{94}Ru			
21	^{98}Cd				

examples of symplectic multiplets formed in the way described above. The nuclei of a given multiplet are uniquely defined by means of N and F_0.

It has been mentioned above that each nucleus corresponds to a subspace of $\mathscr{H}$ where a definite IUR of $u_\pi(6) \oplus u_\nu(6)$ acts. That is why it is reasonable to clarify the sense of the vectors belonging to this subspace. If we assume, in the spirit of IBM-2, that this is the space of the collective states of the nucleus under consideration, then the product $U_\pi(6) \otimes U_\nu(6)$ arises as a DG, and the group $Sp(24, R)$ as a GDG. These exist however,

Table IV. Multiplet (28, 50|50, 82)$_-$

N	F_0: $9/2$	$7/2$	$5/2$	$3/2$	$1/2$	$-1/2$	$-3/2$	$-5/2$
3				^{84}Se	^{84}Ge			
5			^{88}Sr	^{88}Kr	^{88}Se			
7		^{92}Mo	^{92}Zr	^{92}Sr	^{92}Kr			
9	^{96}Pd	^{96}Ru	^{96}Mo	^{96}Zr	^{96}Sr			
11	^{100}Cd	^{100}Pd	^{100}Ru	^{100}Mo	^{100}Zr	^{100}Sr		
13	^{104}Sn	^{104}Cd	^{104}Pd	^{104}Ru	^{104}Mo			
15		^{108}Sn	^{108}Cd	^{108}Pd	^{108}Ru	^{108}Mo		
17			^{112}Sn	^{112}Cd	^{112}Pd	^{112}Ru		
19				^{116}Sn	^{116}Cd	^{116}Pd		
21					^{120}Sn	^{120}Cd		
23						^{124}Sn	^{124}Cd	
25							^{128}Sn	
27								^{132}Sn

Table V. Multiplet (50, 50|82, 82)$_-$

N	F_0 = −1/2	−3/2	−5/2	−7/2	−9/2	−11/2	−13/2	−15/2
3	^{106}Te	^{106}Sn						
5	^{110}Xe	^{110}Te	^{110}Sn					
7		^{114}Xe	^{114}Te	^{114}Sn				
9			^{118}Xe	^{118}Te	^{118}Sn			
11			^{122}Ba	^{122}Xe	^{122}Te	^{122}Sn		
13			^{126}Ce	^{126}Ba	^{126}Xe	^{126}Te	^{126}Sn	
15			^{130}Nd	^{130}Ce	^{130}Ba	^{130}Xe	^{130}Te	^{130}Sn
17			^{134}Sm	^{134}Nd	^{134}Ce	^{134}Ba	^{134}Xe	^{134}Te
19			^{138}Gd	^{138}Sm	^{138}Nd	^{138}Ce	^{138}Ba	
21				^{142}Gd	^{142}Sm	^{142}Nd		
23				^{146}Dy	^{146}Gd			
25				^{150}Er				
27			^{154}Hf					

Table VI. Multiplet (50, 82|82, 126)$_-$

N	F_0 = 11/2	9/2	7/2	5/2	3/2	1/2	−1/2	−3/2	−5/2	−7/2
1						^{134}Te	^{134}Sn			
3					^{138}Ba	^{138}Xe	^{138}Te			
5				^{142}Nd	^{142}Ce	^{142}Ba	^{142}Xe			
7			^{146}Gd	^{146}Sm	^{146}Nd	^{146}Ce	^{146}Ba			
9		^{150}Er	^{150}Dy	^{150}Gd	^{150}Sm	^{150}Nd	^{150}Ce			
11	^{154}Hf	^{154}Yb	^{154}Er	^{154}Dy	^{154}Gd	^{154}Sm	^{154}Nd			
13	^{158}W	^{158}Hf	^{158}Yb	^{158}Er	^{158}Dy	^{158}Gd	^{158}Sm			
15		^{162}W	^{162}Hf	^{162}Yb	^{162}Er	^{162}Dy	^{162}Gd			
17		^{166}Os	^{166}W	^{166}Hf	^{166}Yb	^{166}Er	^{166}Dy			
19		^{170}Pt	^{170}Os	^{170}W	^{170}Hf	^{170}Yb	^{170}Er			
21			^{174}Pt	^{174}Os	^{174}W	^{174}Hf	^{174}Yb			
23			^{178}Hg	^{178}Pt	^{178}Os	^{178}W	^{178}Hf	^{178}Yb		
25				^{182}Hg	^{182}Pt	^{182}Os	^{182}W	^{182}Hf		
27				^{186}Pb	^{186}Hg	^{186}Pt	^{186}Os	^{186}W		
29					^{190}Pb	^{190}Hg	^{190}Pt	^{190}Os	^{190}W	
31						^{194}Pb	^{194}Hg	^{194}Pt	^{194}Os	
33							^{198}Pb	^{198}Hg	^{198}Pt	
35								^{202}Pb	^{202}Hg	
37									^{206}Pb	^{206}Hg

Table VII. Multiplet $(82, 82|126, 126)_-$

	F_0					
N	$-11/2$	$-13/2$	$-15/2$	$-17/2$	$-19/2$	$-21/2$
11	^{186}Pb					
13		^{190}Pb				
15		^{194}Po	^{194}Pb			
17			^{198}Po	^{198}Pb		
19			^{202}Rn	^{202}Po	^{202}Pb	
21			^{206}Ra	^{206}Rn	^{206}Po	^{206}Pb
23				^{210}Ra	^{210}Rn	^{210}Po
25				^{214}Th	^{214}Ra	

some objections against the classification scheme realized above. First, it should be mentioned that nuclei similar in their nature are described by different IURs of $u_\pi(6)\oplus u_\nu(6)$. Thus, if one compares two double magic nuclei belonging to the same multiplet, for instance ^{132}Sn and ^{208}Pb belonging to the multiplet $(50, 82|82, 126)_+$, then it is evident that in the case of ^{132}Sn, N and F_0 are equal to zero and the $u_\pi(6)\oplus u_\nu(6)$ space is one dimensional (it coincides with the vacuum vector in $\mathscr{H}$). At the same time, the nucleus ^{208}Pb is given by $N=38$ and $F_0=-3$ and the corresponding $u_\pi(6)\oplus u_\nu(6)$ space is of a very great dimension. This asymmetry, which leads to the appearance of unphysical states, is avoided in the original version of IBM-2 (Arima *et al.*, 1977), where in the first half of the shell the bosons are

Table VIII. Multiplet $(82, 126|\cdot\cdot)_-$

	F_0					
N	$3/2$	$1/2$	$-1/2$	$-3/2$	$-5/7$	$-7/2$
1		^{210}Po	^{210}Pb			
3	^{214}Ra	^{214}Rn	^{214}Po	^{214}Pb		
5	^{218}Th	^{218}Ra	^{218}Rn	^{218}Po		
7	^{222}U	^{222}Th	^{222}Ra	^{222}Rn		
9		^{226}U	^{226}Th	^{226}Ra	^{226}Rn	
11			^{230}U	^{230}Th	^{230}Ra	
13			^{234}Pu	^{234}U	^{234}Th	
15			^{238}Cm	^{238}Pu	^{238}U	
17		^{242}Fm	^{242}Cf	^{242}Cm	^{242}Pu	^{242}U
19			^{246}Fm	^{246}Cf	^{246}Cm	^{246}Pu
21			^{250}No	^{250}Fm	^{250}Cf	^{250}Cm
23				^{254}No	^{254}Fm	^{254}Cf
25					^{258}No	^{258}Fm

counted as particle pairs and in the second half as hole pairs. On that account, however, there is no way to consider in IBM-2 all even–even nuclei from a given major shell in a unified way.

One possible way to overcome the difficulties connected with the asymmetry, which appears in the $Sp(24, R)$ scheme, is to introduce a proper selection rule for the elimination of the unphysical states. In the next section, concentrating only on the classification of the nuclei, we propose an alternative approach. This approach is based on the group $Sp(4, R)$, which is introduced as a nuclear classification group (CG).

4. $Sp(4, R)$ AS A NUCLEAR CLASSIFICATION GROUP

As mentioned above, the eigenvalues of the operators N and F_0 uniquely determine the nuclei from a given symplectic multiplet. These operators belong to a representation of $sp(4, R)$ [$sp(4, R) \subset sp(24, R)$], which is given by the following generators: $\overset{+}{\pi}_a \overset{+}{\pi}_a$, $\overset{+}{\nu}_a \overset{+}{\nu}_a$, $\overset{+}{\pi}_a \overset{+}{\nu}_a$, $\pi_a \pi_a$, $\nu_a \nu_a$, $\pi_a \nu_a$, $\overset{+}{\pi}_a \pi_a$, $\overset{+}{\nu}_a \nu_a$, $\overset{+}{\pi}_a \nu_a$, $\overset{+}{\nu}_a \pi_a$ (summation over the index a). Hence, the classification problem we are interested in can be associated only with the algebra $sp(4, R)$. The standard boson representation of $sp(4, R)$ can be simply constructed with the help of "one-dimensional" creation (π^+, ν^+) and annihilation (π, ν) operators. The corresponding generators of $Sp(4, R)$ are: $\pi^+\pi^+$, $\nu^+\nu^+$, $\pi^+\nu^-$, $\pi\pi$, $\nu\nu$, $\pi\nu$, $\pi^+\pi$, $\nu^+\nu$, $\pi^+\nu$, $\nu^+\pi$. In other words, further, we do not consider the embedding $sp(4, R) \subset sp(24, R)$. The operators we need now are of the form

$$N_\pi = \pi^+\pi, \qquad N_\nu = \nu^+\nu, \qquad N = N_\pi + N_\nu$$

$$F_+ = \pi^+\nu, \qquad F_- = \nu^+\pi, \qquad F_0 = \tfrac{1}{2}(N_\pi - N_\nu) = \tfrac{1}{2}(C_1^{(1,1)} + 1)$$

where $C_1^{(1,1)}$ is the first Casimir operator of $U(1, 1)$. The space of the boson representation of $sp(4, R)$ will be denoted again by $\mathcal{H}$. This representation splits into two irreducible ones that act in the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$. The structure of $\mathcal{H}_+$ and $\mathcal{H}_-$ is again given by Figures 1a and 1b, respectively, but now the rows represent the IURs of $u(2)$, labeled by N, and the columns represent the IURs of $u(1, 1)$, labeled by F_0. The cells correspond to different IURs of $u_\pi(1) \oplus u_\nu(1)$, defined by (N_π, N_ν) or, which is the same, by (N, F_0).

The physical meaning of the quantities N_π, N_ν, N and F_0 is again given by formulas (4) and (5). Further, the nuclei are arranged in symplectic multiplets [now $sp(4, R)$ multiplets] in the same way as in the case of $sp(24, R)$, described in the previous section, but now each nucleus corresponds to a given IUR of $u_\pi(1) \oplus u_\nu(1)$ [see Tables I–VIII; the notations $(N'_p, N'_n | N''_p, N''_n)_\pm$ are also preserved]. Each row of a given $sp(4, R)$ multiplet contains nuclei belonging to a given $u(2)$ submultiplet, and each column contains nuclei from a given $u(1, 1)$ submultiplet.

In this way the group $Sp(4, R)$ arises as a nuclear classification group (CG). The IURs of $u_\pi(1) \oplus u_\nu(1)$ are one-dimensional, which means that the scheme proposed does not give a possibility for a description of the nuclear states, i.e., the problem of the choice of the DG and GDG, respectively, remains open. On the other hand, now there is no asymmetry as appeared in the case of the $Sp(24, R)$ scheme.

5. 2^+ ENERGY SPECTRUM

The qualitative analysis of the energy spectra of the even–even nuclei with $N_p \geq 20$ and $N_n \geq 20$ (this is the region where the collective effects are rather strongly expressed) reveals the advantages of the $sp(4, R)$ classification scheme proposed above. Figures 2–9 represent the N dependence (at F_0 fixed) of the 2^+ levels of the ground (quasiground) bands for the multiplets given in Tables I–VIII. Here, for the sake of brevity, only multiplets of the type $(N'_p, N'_n | N''_p, N''_n)_-$ (N is odd) are considered. For N even the picture is analogous. The experimental data are from Sakai (1984) and the Nuclear Structure Group (1985/86).

The 2^+ spectra given in Figures 2–9 show the existence of common features, which repeat from shell to shell. The $u(1, 1)$ curves (F_0 fixed) of

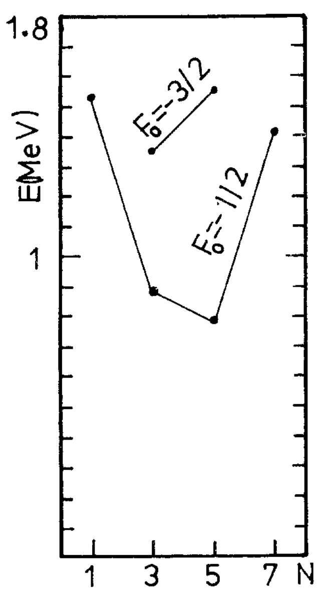


Fig. 2. Multiplet $(20, 20|28, 28)_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table I).

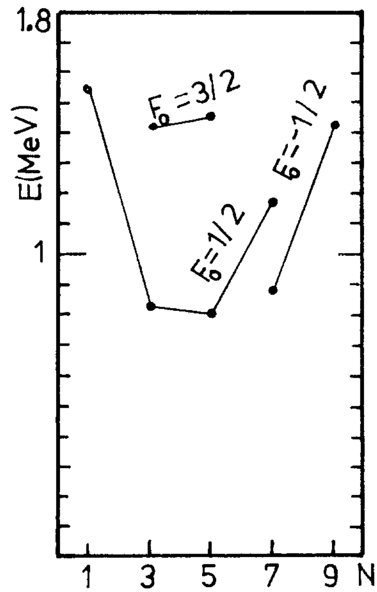


Fig. 3. Multiplet $(20, 28|28, 50)_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table II).

each multiplet are differentiated as a rule and exhibit a similar behavior—the curves increase toward the ends, corresponding to a proton or neutron core, and decrease toward the middle. The similarity of the curves is an indication of the existence of a periodic structure of the shells under consideration. This intrinsic periodic structure is especially stable in the case of the multiplets $(50, 50|82, 82)_-$ (Figure 6), $(50, 82|82, 126)_-$ (Figure 7), and $(82, 126|\cdot\cdot)_-$ (Figure 9) [the same is valid for $(50, 50|82, 82)_+$, $(50, 82|82, 126)_+$, and $(82, 126|\cdot\cdot)_+$]. A strong deviation is observed in the behavior of the curve with $F_0 = 3/2$ of the multiplet $(28, 50|50, 82)_-$ (Figure 5), where the low-lying 2^+ levels of ^{92}Sr ($E_{2^+} = 0.815$ MeV) and especially of ^{96}Zr ($E_{2^+} = 1.751$ MeV) are rather high. This deviation as well as the behavior of the 2^+ spectrum in the region $7 \leq N \leq 11$ of the multiplet $(28, 50|50, 82)_-$ (Figure 5) need special attention. In some sense the 2^+ spectrum of the multiplet $(28, 28|50, 50)_-$ shown in Figure 4 is also anomalous. Analogous anomalies exist in the multiplets $(28, 50|50, 82)_+$ and $(28, 28|50, 50)_+$ as well.

The multiplet $(82, 126|\cdot\cdot)_-$ (Figure 9) contains superheavy nuclei, whose valence nucleons belong to an unclosed major shell. By analogy with the other multiplets, one expects that the 2^+ levels should grow toward the next region of stability. The behavior of the $u(1, 1)$ curves given in Figure 9 shows a relative remoteness from this region—no tendency to any increasing of the curves at higher N is observed.

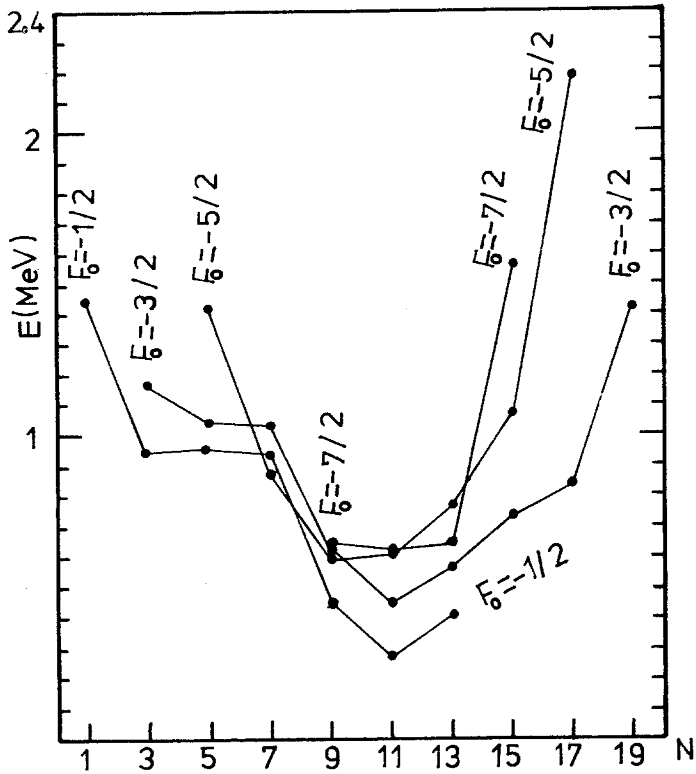


Fig. 4. Multiplet (28, 28|50, 50)$_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table III).

The rotational regions of the multiplets (50, 82|82, 126)$_-$ (Figure 7) and (82, 126|··)$_-$ (Figure 9) are very well expressed. A slight (almost constant) N dependence (at F_0 fixed) of the 2^+ levels is observed in these regions. Another typical feature is the convergence of the $u(1, 1)$ curves of the multiplet (82, 126|··)$_-$ at $N \geq 9$ (Figure 9). The same picture is observed in the rotational regions of the multiplets (50, 82|82, 126)$_+$ and (82, 126|··)$_+$. It should be noted that von Brentano *et al.* (1985) united the rotational nuclei ^{156}Dy-^{184}Hg ($F_0 = 2$) and ^{158}Dy-^{182}Pt ($F_0 = \frac{3}{2}$) from the multiplets (50, 82|82, 126)$_+$ and (50, 82|82, 126)$_-$, respectively, in two F-spin multiplets, where N and F_0 are defined as in case (iii) described in Section 3. Harter *et al.* (1985) united three series of nuclei in F-spin multiplets as follows: ^{124}Te-^{140}Nd with $F_0 = -5$ from (50, 50|82, 82)$_+$, ^{122}Te-^{142}Sm with $F_0 = -9/2$ from (50, 50|82, 82)$_-$ [in both cases N and F_0 are defined as in case (iii) of Section 3], and ^{186}W-^{186}Hg with $N = 27$ from (50, 81|82, 126)$_-$ [N and F_0 are defined as in case (ii) of Section 3].

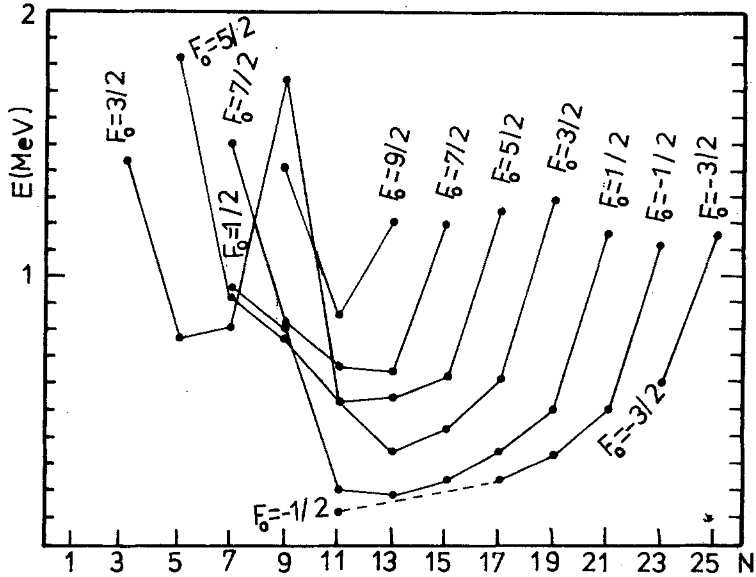


Fig. 5. Multiplet (28, 50|50, 82)$_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table IV).

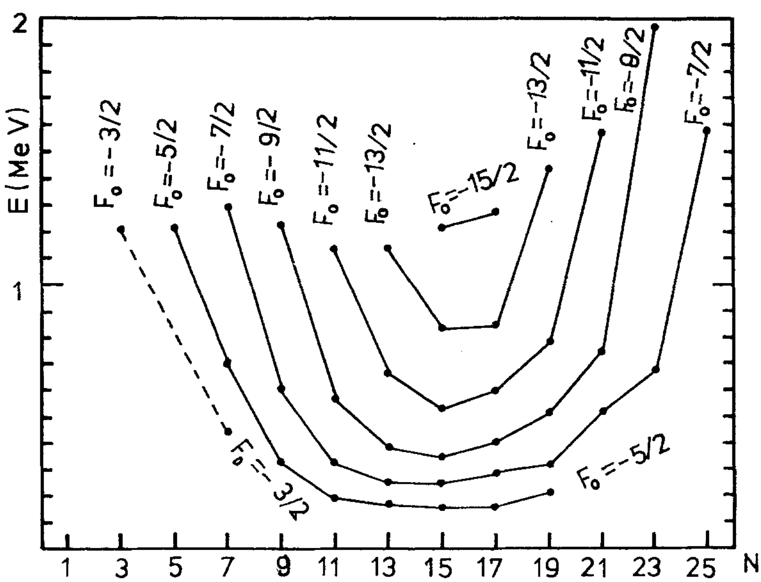


Fig. 6. Multiplet (50, 50|82, 82)$_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table V).

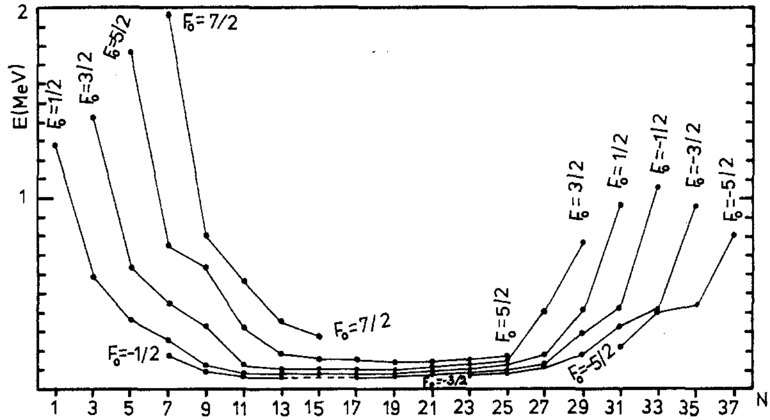


Fig. 7. Multiplet $(50, 82|82, 126)_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table VI).

The neighboring nuclei in the $u(1, 1)$ multiplets differ in an α particle, which is consistent with hypothesis of α clustering in nuclei (Gambhir *et al.*, 1983). A fairly good description of the spectra of the nuclei ^{136}Te–^{168}Er ($F_0 = 0$) from the multiplet $(50, 82|82, 126)_+$ is obtained in a unified way in the framework of the so-called "quartet model" (Dukelsky *et al.*, 1982), where the "quartet bosons" represent quartets of two protons and two neutrons. The quartet effects in the rare-earth nuclei are also considered by Daley *et al.* (1986).

By interpolation we predict the levels (in MeV)

^{162}Gd: $E_{2^+} \approx 0.075$; ^{168}Dy: $0.073 \leq E_{2^+} \leq 0.082$

^{172}Er: $0.073 \leq E_{2^+} \leq 0.082$; ^{234}Pu: $E_{2^+} \approx 0.04$

^{246}Cf: $E_{2^+} \approx 0.043$

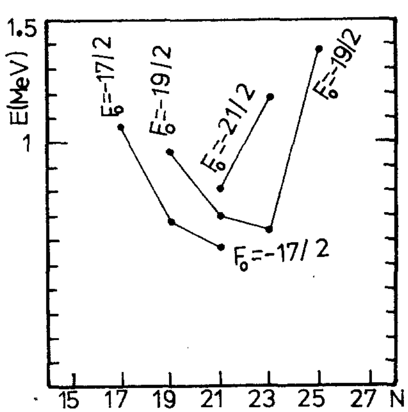


Fig. 8. Multiplet $(82, 82|126, 126)_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table VII).

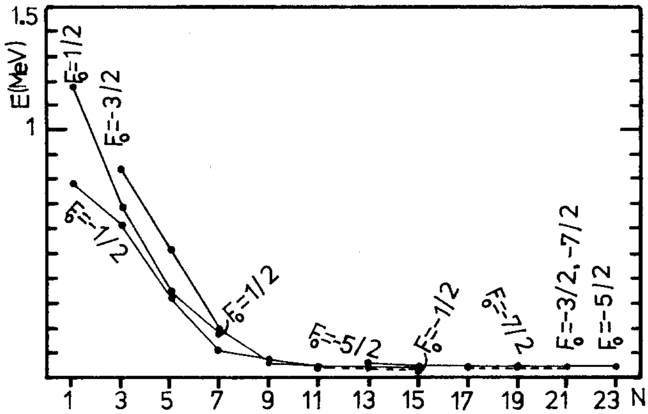


Fig. 9. Multiplet $(82, 126|\cdot\cdot)_-$. Dependence of the 2^+ levels on N at fixed F_0 (see Table VIII).

Thus, the analysis carried out in this paper shows the expediency of the unification of the even–even nuclei with $N_p \geq 20$ and $N_n \geq 20$ in symplectic multiplets. It should be noted that the Hamiltonian, which should describe the energy spectrum of a given symplectic multiplet as a whole, must depend on N_π and N_ν, or, equivalently, on N and F_0. The similarity of the $u(1,1)$ curves [well expressed in the multiplets $(50, 50|82, 82)_\pm$, $(50, 82|82, 126)_\pm$, and $(82, 126|\cdot\cdot)_\pm$] inspires the search of the explicit form of this dependence. This problem will be discussed in a forthcoming paper. Note also that the $u(1,1)$ curves belonging to sequences of $sp(4, R)$ multiplets can be united in common curves under the condition $N_p - N_n$ fixed. Then a periodic variety of the spectrum from one major shell to another is observed.

ACKNOWLEDGMENTS

We are indebted to the members of the seminar of Prof. J. P. Elliott and to Dr. G. Kyrchev for most useful discussions.

This work is supported in part by the Bulgarian Committee of Science, Contract N330.

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